PRIME KNOTS WITH ARC INDEX UP TO 11 AND AN UPPER BOUND OF ARC INDEX FOR NON-ALTERNATING KNOTS

GYO TAEK JIN AND WANG KEUN PARK

Abstract. Every knot can be embedded in the union of finitely many half planes with a common boundary line in such a way that the portion of the knot in each half plane is a properly embedded arc. The minimal number of such half planes is called the arc index of the knot. We have identified all prime knots with arc index up to 11. We also proved that the crossing number is an upperbound of arc index for non-alternating knots. As a result the arc index is determined for prime knots up to twelve crossings.

1. Arc presentations and Arc index

In his foundational work [4] on the *arc index* of knots and links, Cromwell showed that every link diagram is isotopic to a diagram which is a finite union of the following local diagrams in such a way that no more than two corners exist in any vertical line and any horizontal line. Such a diagram is called a *grid diagram*.

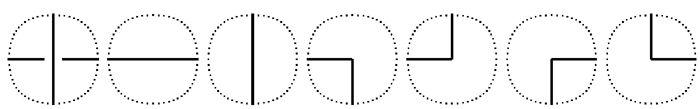

Figure 1. Local parts of diagrams

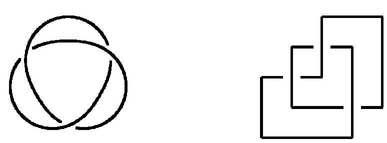

Figure 2. Grid diagram of a trefoil knot

An $n \times n$ matrix each of whose rows and columns has exactly two 1's and 0's elsewhere is called a *Cromwell matrix*. By joining two 1's in each column of a Cromwell matrix with a vertical line segment and two 1's in each row with a horizontal line segment which underpasses any vertical line segments that it crosses, we obtain its grid diagram. Conversely, given a grid diagram with n horizontal lines and n vertical lines, we place 1's at each corner and 0's at other points where the lines and their extensions cross, to construct its Cromwell matrix.

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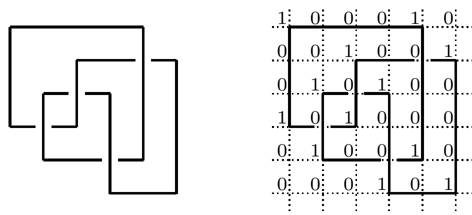


$$\begin{matrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{matrix}$$

FIGURE 3. Construction of Cromwell matrix from a grid diagram and its inverse

Proposition 1 (Cromwell 1995)**.** *Every link admits an arc presentation.*

Proof. As illustrated in Figure 4, the horizontal line segments of a grid diagram are horizontally pulled backwards to touch a vertical axis behind the diagram to form an arc presentation. As every link has a grid diagram, we can conclude that every link has an arc presentation. □

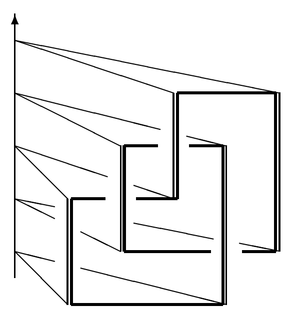


FIGURE 4. Construction of an arc presentation from a grid diagram

The minimal number of pages of all arc presentations of a link L is called the *arc index* of L and is denoted by $\alpha(L)$. It is known that the trefoil knot is the only link with arc index 5. The table below shows all links up to arc index 5. In [14], Nutt identified all knots up to arc index 9. In [3], Beltrami determined arc index for prime knots up to ten crossings. In [8], Jin et al. identified all prime knots up to arc index 10. Ng determined arc index for prime knots up to eleven crossings [12]. Matsuda determined the arc index for torus knots [10]. The *Table of Knot Invariants* [18] gives the arc index for prime knots up to 12 crossings.

TABLE 1. Links with arc index up to 5

$\alpha(L)$	2	3	4	5
L	unknot	none	2-component unlink, Hopf link	trefoil

2. TABULATION OF PRIME KNOTS BY ARC INDEX

As there are finitely many $n \times n$ Cromwell matrices for each $n \geq 2$, there are finitely many knots with arc index up to n, for each $n \geq 2$, hence, there are only finitely many knots with arc index n, for each $n \geq 2$. By the following theorem

of Cromwell, we only need to tabulate arc index of prime knots to determine arc index of all knots.

Theorem 2 (Cromwell 1995). *If K_1, K_2 are nontrivial, then*

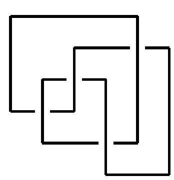
$$\alpha(K_1 \sharp K_2) = \alpha(K_1) + \alpha(K_2) - 2$$

The following moves on a Cromwell matrix does not change the link type of the corresponding grid diagram up to mirror images.

M1. Flipping in a horizontal axis, a vertical axis a diagonal axis and an antidiagonal axis.
M2. Rotation in the plane by 90 degrees.
M3. Moving the first row to the bottom and moving the first column to the rear.
M4. Exchange of two adjacent rows or columns whose ones are in non-interleaving position.

$$\begin{bmatrix} & & & \cdots & & & \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 \\ & & & \cdots & & & \end{bmatrix} \Longleftrightarrow \begin{bmatrix} & & & \cdots & & & \\ 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ & & & \cdots & & & \end{bmatrix}$$

FIGURE 5. Interchange of non-interleaved rows



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

FIGURE 6. $100010\ 001001\ 010100\ 101000\ 010010\ 000101_2$

The *norm* of a Cromwell matrix is the n^2 digit binary number obtained by concatenating its rows. An example is shown in Figure 6.

To tabulate prime knots with arc index up to 11, we proceeded with the following steps, for each integer $n = 5, \ldots, 11$.

(1) Generate all $n \times n$ Cromwell matrices whose leading entry is 1 in the norm-decreasing order.
(2) Discard those corresponding to links of more than one components.
(3) Discard if its grid diagram is not prime.
(4) Discard if a sequence of moves M1–M4 ever increase the norm.
(5) Discard if a sequence of move M4 ever makes two 1's adjacent horizontally or vertically, as their existence causes a reduction of the size of Cromwell matrix.
(6) Identify the knot of its grid diagram.
(7) Discard the knot if it already appeared for a smaller n.

Our computer program in which the steps (1)–(5) were implemented produced 663,341 Cromwell matrices for $n = 11$ and their Dowker-Thistlethwaite codes ('DT codes' for short). Using Knotscape [17], we were able to eliminate most of the duplications and obtained 2,721 distinct DT codes of prime knots. They include

Table 2. Prime knots up to arc index 11 or up to 12 crossings

Arc index / Crossings	5	6	7	8	9	10	11	12	13	14	Subtotal
3	1^*										1
4		1^*									1
5			2^*								2
6				3^*							3
7					7^*						7
8			1	2		18^*					21
9				2	6		41^*				49
10				1	9	32		123^*			165
11					4	46	135		367^*		552
12					2	48	**211**†	**627**		1288^*	2176
13						49	**399**				
14						17	**477**				
15					1	22	**441**				
16						7	**345**				
17						1	**192**				
18							**75**‡				
19							**12**‡				
20							**3**‡				
21							**3**‡				
22											
23											
24							**1**				
Subtotal	1	1	3	8	29	240	**2335**				

- All prime knots of arc index 11. There are 2,335 such knots.
- All prime knots up to arc index 10 except $13n_{4639}$. There are 281 such knots [8].
- All prime knots up to 10 crossings except the 10 crossing alternating knots. There are 126 such knots.
- Some duplications on knots with more than 16 crossings.

To handle the duplications on knots with more than 16 crossings, we used polynomial invariants and hyperbolic invariants built into Knotscape. We also used Knotplot [16] to isotope a knot for another DT code that may work better in Knotscape. Alexander Stoimenow strained out all the duplicates which we could not handle. The knots of arc index 11 counted in Table 2 are all distinct although the crossing number may change for those of 18 crossings or higher. The authors

*The number of prime alternating knots with the given crossing number.
†The numbers in boldface are newly determined by this work.
‡Some of these knots may have smaller crossing number but not smaller than 17.

wrote a separate article for the list of all prime knots with arc index up to 11 and their minimal arc presentations [9].

3. Alternating knots and non-alternating knots

Combining Theorem 3 and Theorem 4, we know that the arc index of alternating links is equal to the minimal crossing number plus 2. The main diagonal of Table 2 shows this fact. The blanks below the main diagonal of Table 2 verifies that the inequality in Theorem 5 holds up to 12 crossing knots. Theorem 3 was conjectured by Cromwell and Nutt [5]. Beltrami obtained an inequality sharper than the one in Theorem 5 for semi-alternating links [3].

Theorem 3 (Bae-Park)**.** *Let L be any prime link and let $c(L)$ denote the minimal crossing number of L. Then $\alpha(L) \leq c(L) + 2$.*

Theorem 4 (Morton-Beltrami)**.** *Let L be an alternating link with the minimal crossing number $c(L)$, then $\alpha(L) \geq c(L) + 2$.*

Theorem 5. *A prime link L is non-alternating if and only if*

$$\alpha(L) \leq c(L).$$

This theorem allows us to put 627 in Table 2 for the number of 12 crossing knots with arc index 12.

4. Proof of Theorem 5

A *knot-spoke diagram* D is a finite connected plane graph with the following properties.

(1) There are three kinds of vertices in D; a distinguished vertex v_0 with valency at least four, 4-valent vertices, and 1-valent vertices.
(2) Every edge incident to a 1-valent vertex is also incident to v_0. Such an edge is called a *spoke*.

A knot-spoke diagram D is said to be *prime* if no simple closed curve meeting D in two interior points of edges separates multi-valent vertices into two parts. A multi-valent vertex v of a knot-spoke diagram D is said to be a *cut-point* if there is a simple closed curve S meeting D in v and separating non-spoke edges into two parts. A cut-point free knot-spoke diagram with more than one non-spoke edges cannot have a *loop**. A loop in D is said to be *simple* if the other non-spoke edges are in one side. If a prime knot-spoke diagram D has a cut-point, then the distinguished vertex v_0 must be the cut-point with valency bigger than four. A knot-spoke diagram without any non-spoke edges is called a *wheel diagram*.

The valency of the distinguished vertex v_0 is an even number plus the number of spokes. To obtain the type of a knot or link which can be projected onto a knot-spoke diagram D, we may assign relative heights of the endpoints of edges of D in the following way.

(3) At every 4-valent vertex, pairs of opposite edges meet in two distinct levels. Their relative heights are indicated by drawing a small neighborhood of the vertex as a crossing in a knot diagram.

*A closed curve created by a single edge

(4) If the distinguished vertex v_0 is incident to $2a$ non-spoke edges and b spokes, then its small neighborhood is the projection of $n = a + b$ arcs at distinct levels whose relatives heights can be specified by the numbers $1, \cdots, n$. Every spoke is understood as the projection of an arc on a vertical plane whose endpoints project to v_0.

Suppose a knot-spoke diagram D has height information at multi-valent vertices, then

(5) D determines a knot(or link) L. In this case, we say that D is a knot-spoke diagram of L.
(6) If D has only 4-valent vertices then it is a knot(or link) diagram.
(7) If D has no non-spoke edges, it is an arc presentation.

In [1], Bae and Park† used knot-spoke diagrams to prove Theorem 3.

Let e be an edge of a cut-point free knot-spoke diagram D incident to v_0 and to another vertex v_1 which is 4-valent. We denote by D_e the knot-spoke diagram obtained by contracting e and replacing any simple loop thus created by a spoke in the following way:

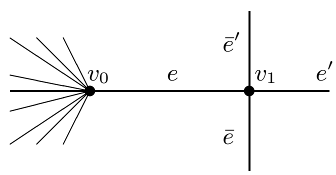


FIGURE 7. Local diagram of D near e

Suppose that e, $\bar{e}$, e' and $\bar{e}'$ are four edges incident to v_1, around in this order so that v_1 is the crossing of the two arcs $e \cup e'$ and $\bar{e} \cup \bar{e}'$. Before the contraction of e, we adjust the endpoint-heights at v_1 as follows.

(8) e is made horizontal so that e' is given the height of e at v_0.
(9) If $\bar{e} \cup \bar{e}'$ undercrosses $e \cup e'$ then $\bar{e}$ and $\bar{e}'$ are given a height lower than the lowest height at v_0. If $\bar{e} \cup \bar{e}'$ overcrosses $e \cup e'$ then $\bar{e}$ and $\bar{e}'$ are given a height higher than the highest height at v_0.

Then as e is contracted to v_0, the ends of $\bar{e}$, e' and $\bar{e}'$ at v_1 are moved to v_0 horizontally along e. If none of $\bar{e}$, e' and $\bar{e}'$ is incident to v_0 in D, no loop is created by the contraction of e. In this case, D_e is obtained by the contraction of e only. Suppose $\bar{e}$ is incident to v_0 in D. Because D is cut-point free, $\bar{e}$ becomes a simple loop by the contraction of e. To replace this loop by a spoke, we fold the half of $\bar{e}$ previously incident to v_1 onto the other half. If in case $\bar{e}'$ is incident to v_0 in D, a simple loop is created and is replaced by a spoke in a similar manner. If in case e' is incident to v_0 in D, a non-simple loop is created. This loop remains as is in D_e so that v_0 becomes a cut-point unless the loop is the only non-spoke edge of D_e.

There are two important facts to point out.

(10) Under the process of of creating D_e from D, the sum of the number of regions divided by the diagram and the number of spokes is unchanged.
(11) D_e is prime if D is prime.

†Not the second author of this article

Lemma 6. *Let D be a knot-spoke diagram without cut-points. Suppose that D has at least two multi-valent vertices. Then there are at least two non-loop non-spoke edges e and f, incident to v_0, such that the knot-spoke diagrams D_e and D_f have no cut-points.*

This lemma is adapted from Lemma 7 [1, Lemma 1] to apply directly to knot-spoke diagrams. We omit the proof as this will be almost a direct translation into the language of knot-spoke diagrams.

Lemma 7 (Bae-Park)**.** *Let G be a connected plane graph without cut vertices and v a vertex of G. For an edge e incident to v, let $\overline{G_e}$ denote the graph obtained from G by contracting e and then by deleting all the innermost loops based at v. Suppose that G has at least two vertices and all vertices except v are 4-valent. Then there are at least two edges e and f, incident to v, such that new graphs $\overline{G_e}$ and $\overline{G_f}$ have no cut vertices.*

Suppose that D is a connected link diagram of a link L without nugatory crossings. Let $c(D)$ and $r(D)$ denote the number of crossings in D and the number of regions of the plane divided by D, respectively. Then $r(D) = c(D) + 2$. In proving Theorem 3, the authors of [1] considered D as a knot-spoke diagram and showed that there is a sequence of edges $e_1, \dots, e_k$ of D such that

$$D_{e_1 \cdots e_k} = (\cdots (D_{e_1})_{e_2} \cdots)_{e_k}$$

has only one non-spoke edge which then must be a loop. Using the fact

$$s(D_{e_1 \cdots e_i}) + r(D_{e_1 \cdots e_i}) = r(D)$$

for all i, where $s(\ )$ denotes the number of spokes, they found that $s(D_{e_1 \cdots e_k}) = r(D) - 2 = c(D)$. Then the loop which is the only non-spoke edge can be made into two spokes so that the wheel diagram so obtained has exactly $c(D)+2$ spokes, showing $\alpha(L) \leq c(D)+2$. The inequality $\alpha(L) < c(D)+2$ when D is non-alternating was obtained by contracting two adjacent edges simultaneously creating one less spokes than usual [1, 3]. In the following, we argue that such contractions can be done twice, simultaneously or successively, to construct a wheel diagram having $c(D)$ spokes, showing $\alpha(L) \leq c(L)$ when D has the minimal number of crossings.

Before we analyze non-alternating diagrams for our purpose, we state our main tool.

Proposition 8. *Let D be a prime cut-point free knot-spoke diagram and let e be an edge incident to v_0 and to another 4-valent vertex v_1 such that D_e has a cut-point. Then there exists a simple closed curve S_e satisfying the following conditions.*

1. $D_e \cap S_e = v_0$
2. *S_e separates $\bar{e}$ and $\bar{e}'$ where the four edges incident to v_1 in D are labeled with e, $\bar{e}$, e', $\bar{e}'$ as in Figure 7.*
3. *S_e separates D_e into two knot-spoke diagrams $\bar{D}$ and $\bar{D}'$ containing $\bar{e}$ and $\bar{e}'$, respectively. Furthermore $\bar{D}'$ is prime and cut-point free, and there is a sequence of non-spoke edges $e_1, \dots, e_k$ of D not contained in $\bar{D}'$ such that the knot-spoke diagram $D_{e_1 e_2 \cdots e_k}$ is identical with $\bar{D}'$ on non-spoke edges in one side of S_e and has only spokes in the other side.*

Proof. Because D is cut-point free, v_0 must be the only cut-point of D_e, hence a simple closed curve S_e satisfying (1) exists. Then we may consider a simple closed

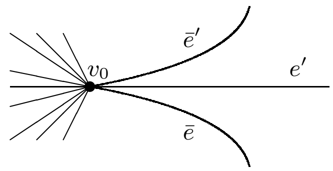


FIGURE 8. Local diagram of D_e near v_0

curve S such that $D \cap S = e$ and S_e is obtained from S by contracting e. If S_e does not separate $\bar{e}$ and $\bar{e}'$ in D_e, then S can be pushed off e so that it meets D only in v_0. This contradicts the assumption that D is cut-point free. Therefore condition (2) holds. Because D is cut-point free, we only need to consider the following three cases to prove condition (3).

- *Case 1.* The S_e satisfying (1) and (2) is essentially unique and S separates $\{\bar{e}, e', \bar{e}'\}$ into $\{\bar{e}\}$ and $\{e', \bar{e}'\}$.
- *Case 2.* The S_e satisfying (1) and (2) is essentially unique and S separates $\{\bar{e}, e', \bar{e}'\}$ into $\{\bar{e}, e'\}$ and $\{\bar{e}'\}$.
- *Case 3.* There are two essentially unique distinct simple closed curves S and S' satisfying $D \cap S = D \cap S' = e$ such that S_e and S'_e satisfy (1) and (2), and S and S' separate $\{\bar{e}, e', \bar{e}'\}$ into $\{\bar{e}\}$, $\{e', \bar{e}'\}$ and $\{\bar{e}, e'\}$, $\{\bar{e}'\}$, respectively.

In the Figures 9–19 of knot-spoke diagrams, all the spokes are omitted for simplicity.

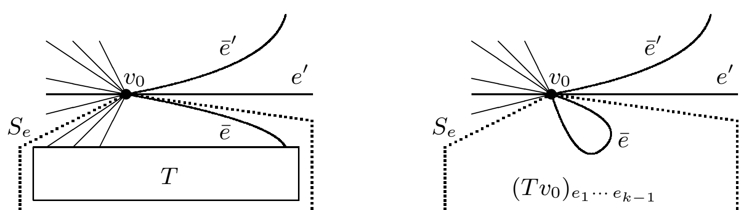


FIGURE 9. Case 1 : D_e and $D_{e\, e_1 \cdots e_{k-1}}$

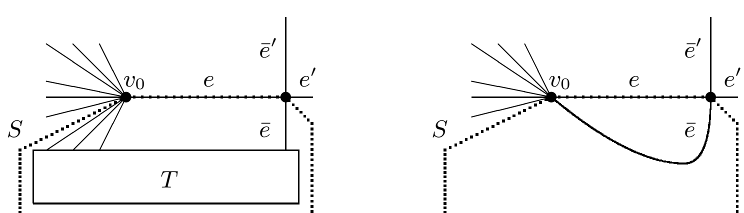


FIGURE 10. D and $D_{e_1 \cdots e_{k-1}}$

Case 1. As shown in Figure 9, we consider the tangle T inside S_e whose end arcs are joined to v_0 to make a knot-spoke diagram Tv_0. Then Tv_0 is cut-point free. By applying Lemma 6 repeatedly to Tv_0, there is a sequence of edges $e_1, \ldots, e_{k-1}$

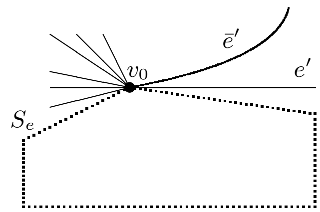


FIGURE 11. $D_{e_1 \cdots e_k}$

of D inside S (hence of Tv_0), all distinct from $\bar{e}$, such that the knot-spoke diagram $(Tv_0)_{e_1 \cdots e_{k-1}}$, created inside S_e, has only one non-spoke edge which is the loop corresponding to $\bar{e}$. As shown in Figure 10, in the knot-spoke diagram $D_{e_1 \cdots e_{k-1}}$, the two edges e and $\bar{e}$ together bound a region which may contain spokes. Taking $e_k = e$, the knot-spoke diagram $D_{e_1 \cdots e_k}$ obtained by contracting e and replacing $\bar{e}$ by a spoke is identical with D_e outside S_e and has only spokes inside S_e. Therefore condition (3) is satisfied in this case.

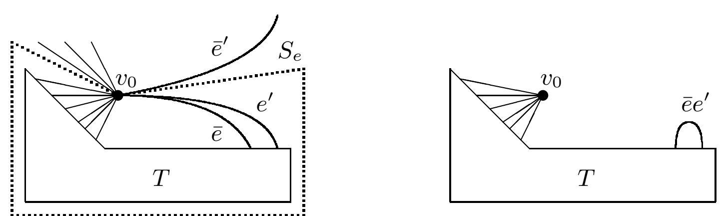


FIGURE 12. Case 2 : D_e and $\bar{T}v_0$

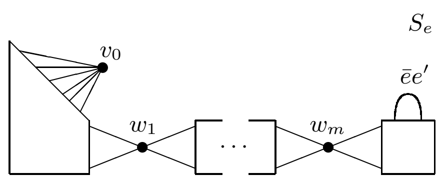


FIGURE 13. $\bar{T}v_0$ may have cut-points $w_1, \ldots, w_m$.

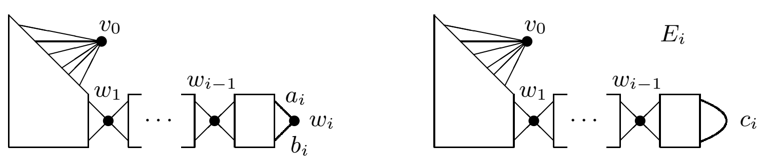


FIGURE 14. E_i is obtained by truncating $\bar{T}v_0$ at w_i.

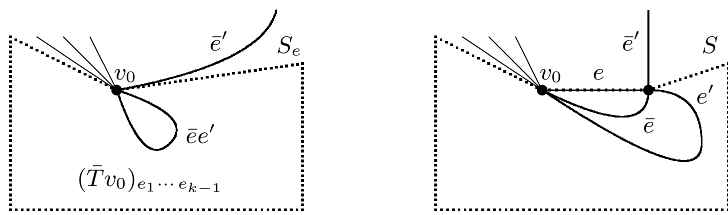


FIGURE 15. $(\bar{T}v_0)_{e_1\cdots e_{k-1}}$ and $D_{e_1\cdots e_{k-1}}$

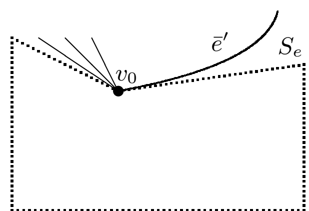


FIGURE 16. $D_{e_1\cdots e_k}$

Case 2. As in Figure 12, in the knot-spoke diagram Tv_0, the two edges $\bar{e}$ and e' are adjacently incident to v_0. We take off $\bar{e}$ and e' from v_0 and create a new edge $\bar{e}e'$. Then we have a knot-spoke diagram $\bar{T}v_0$ inside S_e which may have cut-points $w_1, \dots, w_m$ as shown by Figure 13. For $i = 1, \dots, m$, let E_i be the knot-spoke diagram obtained by truncating $\bar{T}v_0$ at the cut-point w_i and amalgamating two edges a_i and b_i into c_i. By applying Lemma 6 to E_1, we obtain a sequence $e_1, \dots, e_{k_1-1}$ of edges of E_1, all distinct from c_1, such that $(E_1)_{e_1\cdots e_{k_1-1}}$ has only one non-spoke edge which is a loop corresponding to c_1. Taking $e_{k_1} = a_1$, the knot-spoke diagram $(E_2)_{e_1\cdots e_{k_1}}$ is obtained from $(E_2)_{e_1\cdots e_{k_1-1}}$ by contracting a_1 and replacing the loop b_1 by a spoke. Proceeding in this manner, we obtain a sequence $e_{k_i+1}, \dots, e_{k_{(i+1)}-1}, e_{k_{(i+1)}} = a_{i+1}$ of edges of $E_{i+1} \setminus E_i$ such that $(E_{i+1})_{e_1\cdots e_{k_{(i+1)}-1}}$ has only one non-spoke edge which is a loop corresponding to c_{i+1}, for $i = 1, \dots, m-1$. Taking $e_{k_m} = \bar{e}$, the knot-spoke diagram $(\bar{T}v_0)_{e_1\cdots e_{k_m}}$ is cut-point free. Applying Lemma 6, we obtain a sequence $e_{k_m+1}, \dots, e_{k-1}$ of edges of $\bar{T}v_0 \setminus E_m$ such that the knot-spoke diagram $(\bar{T}v_0)_{e_1\cdots e_{k-1}}$ has only one non-spoke edge corresponding to the loop $\bar{e}e'$. Finally, by taking $e_k = \bar{e}$, the knot-spoke diagram $D_{e_1\cdots e_k}$ obtained by contracting $\bar{e}$ and replacing the loops corresponding to e and e' by spokes, has no non-spoke edges inside S_e and no non-spoke edges outside S_e has been contracted or became a spoke except e. Therefore condition (3) is satisfied in this case.

Case 3. Since S_e and S'_e meet in v_0, they divide the plane into two bounded regions and one unbounded region. We may assume that S'_e is the boundary of the unbounded region. Then we have two cut-point free knot-spoke diagrams Tv_0 and $T'v_0$ inside S_e and between S_e and S'_e, respectively, as in Figure 17. By applying Lemma 6 to Tv_0, we obtain a sequence $e_1, \dots, e_j$ of edges of Tv_0 such that $(Tv_0)_{e_1\cdots e_j}$ has only one non-spoke edge which is a loop corresponding to $\bar{e}$. By applying Lemma 6 to $T'v_0$, we obtain a sequence $e_{j+1}, \dots, e_{k-1}$ of edges of $T'v_0$ such that $(T'v_0)_{e_{j+1}\cdots e_{k-1}}$ has only one non-spoke edge which is a loop

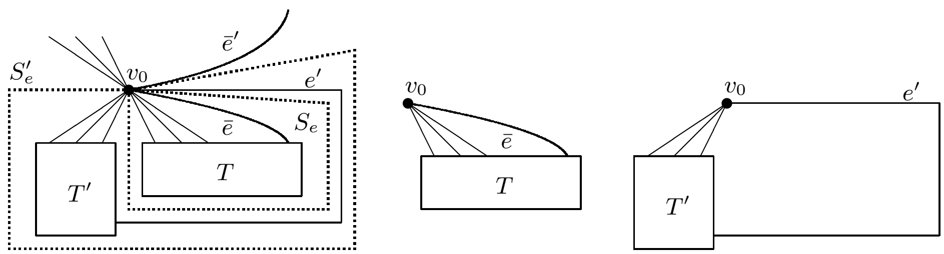


FIGURE 17. Case 3: D_e, Tv_0 and $T'v_0$

corresponding to e'. Then

$$(Tv_0)_{e_1\cdots e_j} \cup (T'v_0)_{e_{j+1}\cdots e_{k-1}} = (Tv_0 \cup T'v_0)_{e_1\cdots e_{k-1}}.$$

Finally, by taking $e_k = \bar{e}$, the knot-spoke diagram $D_{e_1\cdots e_k}$ obtained by contracting $\bar{e}$ and replacing the loops corresponding to e and e' by spokes, has no non-spoke edges inside S'_e and no non-spoke edges outside S'_e has been contracted or became a spoke except e. Therefore condition (3) is satisfied in this case. □

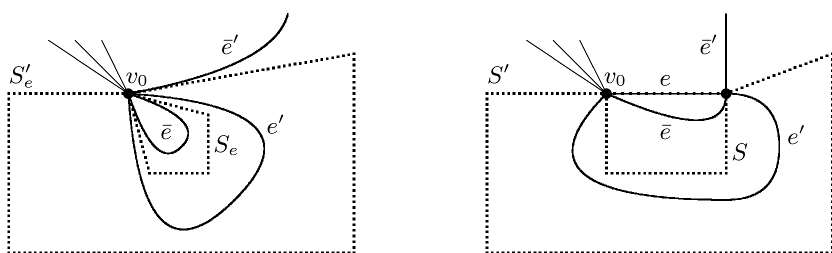


FIGURE 18. $(Tv_0 \cup T'v_0)_{e_1\cdots e_{k-1}}$ and $D_{e_1\cdots e_{k-1}}$

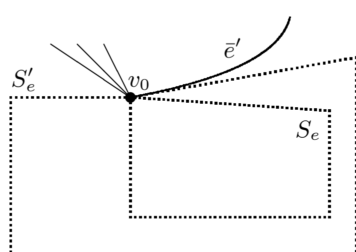


FIGURE 19. $D_{e_1\cdots e_k}$

Suppose that D is a prime non-alternating minimal crossing link diagram. An innermost or outermost region of the plane bounded by edges of D is called an *alternating region* of D if its boundary edges are all alternating and a *non-alternating region* of D otherwise. Let $D^\star$ denote the subgraph of the dual graph of D whose edges are the duals of non-alternating edges of D. One should notice that among the non-alternating edges, the undercrossing ones and the overcrossing ones appear alternatingly around any non-alternating region. This shows that, in $D^\star$, every

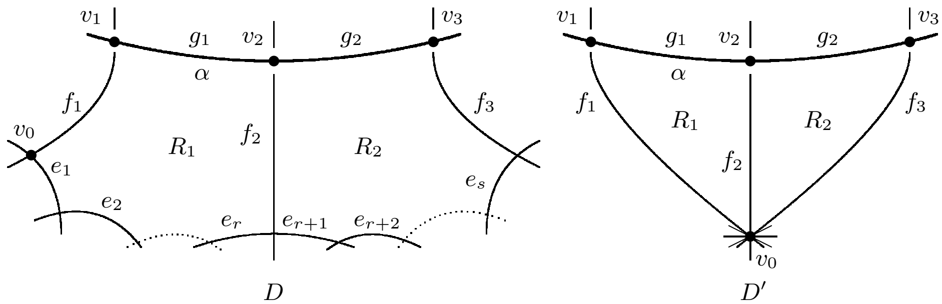


FIGURE 20. Case I : D and D'

vertex is even-valent, every edge belongs to a cycle, and every cycle has an even number of edges. Because D is prime, no bigons exist in $D^\star$. Now we claim that at least one of the three following cases occur.

- *Case I.* There is an arc of D which crosses over (or under) three times consecutively.
- *Case II.* The two edges of an arc crossing one end of a non-alternating edge are both alternating.
- *Case III.* One of the two edges of an arc crossing one end of a non-alternating edge is alternating and the other is non-alternating.

Suppose that an innermost cycle C in $D^\star$ does not contain any alternating region of D. Then the part of D inside C is a tree T consisting of edges and half edges. There is a vertex v in T where at least three half edges are incident. The extension of the two nonadjacent half edges is an arc which has three consecutive overcrossings or undercrossings. Suppose that an innermost cycle C in $D^\star$ contains an alternating region, then, because non-alternating regions cannot be isolated, there exists a vertex at which at least one alternating region and at least two non-alternating regions meet. This proves our claim.

Case I. Suppose there is an arc α of D consecutively crossing over (resp. under) at the vertices v_1, v_2 and v_3. For $i = 1, 2$, let g_i be the edge joining v_i and v_{i+1}. On one side of this arc there are two adjacent non-alternating regions R_1 and R_2, whose boundary edges are $f_1, e_1, \ldots, e_r, f_2, \overline{v_1v_2}$ and $f_2, e_{r+1}, \ldots, e_s, f_3, \overline{v_2v_3}$, respectively, as shown in Figure 20. Let v_0 be the vertex which is the common endpoint of f_1 and e_1. By contracting the edges $e_1, \ldots, e_s$ to v_0 one after another, we can obtain a new knot-spoke diagram $D_{e_1\cdots e_s}$ in which the regions R_1, R_2 become triangular. If v_0 is a cut-point, then find i such that e_i is the first edge that makes v_0 into a cut-point of $D_{e_1\cdots e_i}$, then there exists a simple closed curve S_{e_i} meeting $D_{e_1\cdots e_i}$ in v_0 having the regions R_1 and R_2 in one side. We apply Proposition 8 to deform the non-spoke edges in the other side of S_{e_i} either by contracting or by making into spokes. We can repeat this process until we obtain a cut-point free knot-spoke diagram D' in which R_1 and R_2 are triangular as in Figure 20. So far we haven't changes the sum of the number of regions and the number of spokes, hence

$$s(D') + r(D') = r(D) = c(D) + 2.$$

Let D'' be obtained from D' by isotoping the arc $\alpha = g_1 \cup g_2$ over (resp. under) the vertex v_0. This is the same as contracting the edges f_1, f_2, f_3 simultaneously and

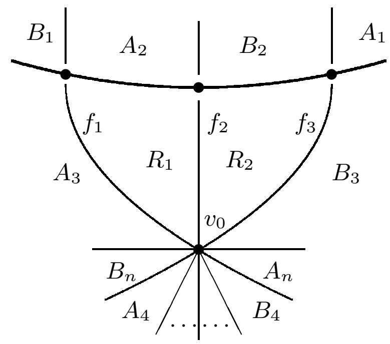


FIGURE 21. Regions around $R_1 \cup R_2$ in D'

then shrinking the two loops obtained by the overcrossing (resp. undercrossing) arc α to a point above (resp. below) v_0. This is possible because α can be placed above (resp. below) any other edges incident to v_0. Then D'' is a knot-spoke diagram satisfying

$$s(D'') + r(D'') = r(D) - 2 = c(D). \tag{\dagger}$$

Unless D'' has a cut-point, we are done in this case.

Suppose that D'' has a cut-point. As the cut-point must be the vertex v_0 and the local diagram around $R_1 \cup R_2$ in D' is symmetric, we only need to check the cases when two regions A_i and A_j, labeled as in Figure 21, indicate the same region, for $i \neq j$. Because D is a prime minimal crossing diagram, we do not need to consider the cases $A_i = A_{i+1}$ for $i = 1, \ldots, n-1$, or $A_1 = A_n$. For the case $A_1 = A_n$, it may happen that A_1 has an edge which form a bigon region with f_3. This edge is replaced by a spoke when f_3 is contracted. Because D' is cut-point free, we have $A_i \neq A_j$ for $3 \leq i < j \leq n$. Therefore the nontrivial cases occur only if $n \geq 4$, and they are

(1) $A_1 = A_i$ or $B_1 = B_i$ for $i = 3, \ldots, n-1$,
(2) $A_2 = A_i$ or $B_2 = B_i$ for $i = 4, \ldots, n$.

If case (1) occurs only, we can apply Proposition 8 to D outside the region $R_1 \cup R_2$ to obtain a cut-point free knot-spoke diagram D'' satisfying the equation (†). If case (2) occurs, there is a simple closed curve S meeting D in three points which are a point on g_1, a point on f_2, and a vertex of ∂R_2 not incident to f_2 or f_3. Contracting

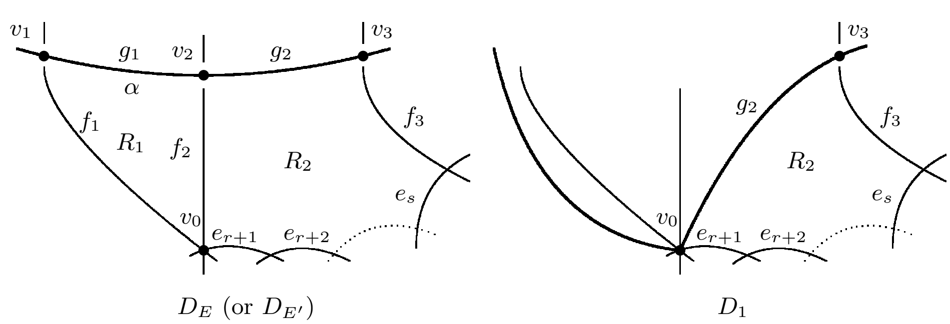


FIGURE 22. Case I (2)

the edges $e_1, \ldots, e_r$ and applying Proposition 8 if necessary, we obtain a sequence E of edges to obtain a cut-point free knot-spoke diagram D_E, as illustrated in Figure 22, in which R_1 is triangular and R_2 is as in D. Isotoping g_1 over v_0 and replacing any simple loops created by this isotopy with spokes, we obtain a knot-spoke diagram D_1 satisfying

$$s(D_1) + r(D_1) = r(D) - 1 = c(D) + 1. \tag{‡}$$

If v_0 is a cut-point of D_1, then it must be case (1), and hence we can apply Proposition 8 to obtain another sequence of edges E' so that the regions R_1 and R_2 in $D_{E'}$ are as in D_E. After the same isotopy of g_1, we obtain a cut-point free knot-spoke diagram, again denoted by D_1, satisfying equation (‡). Contracting the edges $e_{r+1}, \ldots, e_s, f_3$, one by one, we obtain a spoke of $(D_1)_{e_{r+1}\cdots e_s f_3}$ created by the arc g_2 which can be placed above (resp. below) any other edges nearby. This spoke can be shrunk to a point above (resp. below) any other edges at v_0, creating a knot-spoke diagram D'' satisfying equation (†). In the case D'' has a cut point, we can apply Proposition 8 to obtain another sequence F of edges creating an uppermost (resp. lowermost) spoke created by g_2 in the cut-point free knot-spoke diagram D''_F. Shrinking the spoke, we obtain a cut-point free knot-spoke diagram, again denoted by D'' satisfying equation (†).

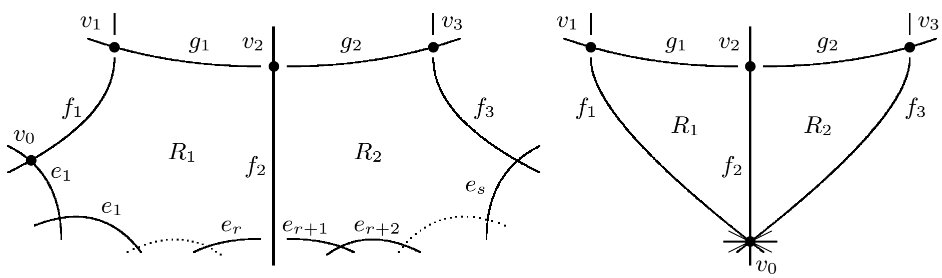
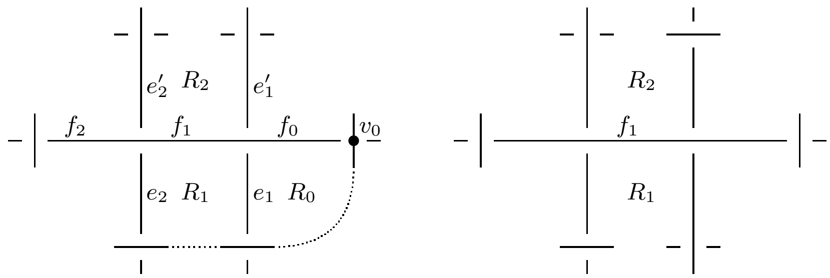


FIGURE 23. Case II

Case II. As shown in Figure 23, two non-alternating regions R_1, R_2 are adjacent along a non-alternating edge f_2 and the edges g_1, g_2 belonging to $\partial R_1, \partial R_2$, respectively, are incident to the endpoint v_2 of f_2. We may assume that there are two horizontal planes P and Q such that P separates f_2 from the other edges of ∂R_1 and ∂R_2 and Q separates f_2, g_1, g_2 from the rest. The same edge contractions and isotopies as in *Case I* can be done within the respective parts divided by P and Q to obtain a cut-point free knot-spoke diagram D'' satisfying equation (†).

Case III. If *Case I* and *Case II* do not occur in D, then, as illustrated in Figure 24, there exist only two distinct patterns for the types of the edges incident to a non-alternating edge f_1 where two regions R_1 and R_2 are incident. In the first pattern, we perform edge contractions to make R_0 into a triangular region and then isotope e_1 to eliminate the region R_0. We do further edge contractions to make R_1 into a triangular region and then isotope e_2 to eliminate R_1. If v_0 is the cut-point of the knot-spoke diagram just obtained, then applying Proposition 8 we can do the edge contractions all over again so that the elimination of R_0 and R_1 is done by isotoping e_1 and e_2 without making v_0 into a cut-point unless there exists a simple closed curve S meeting D at a point of e_2, at a point of e_1 and at a vertex of ∂R_0 away



(1) (2)

FIGURE 24. Case III

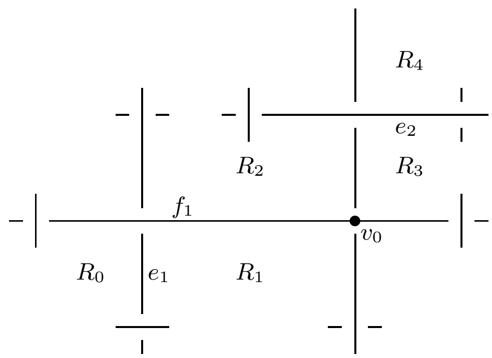


FIGURE 25. Case III (2)

from the edge e_1. In this exceptional case we can also apply Proposition 8 so that the first reduction of $s(D)+r(D)$ is done by an isotopy of e_1 and the second is done by contracting e_2 (or e_2') after it become incident to v_0. This contraction makes f_1 into a loop which can be kept in a horizontal level in order to be isotoped off. In this way we obtain a cut-point free knot-spoke diagram D'' satisfying equation (†).

By assuming that *Case I*, *Case II* and *Case III* (1) do not occur anywhere in D, the second pattern extends as shown in Figure 25. We may attempt to eliminate the regions R_1 and R_3 by isotoping e_1 and e_2 after some edge contractions. The only nontrivial case that v_0 becomes a cut-point occurs when there exists a simple closed curve S meeting D in four points including a point in e_1 and a point in e_2. In this case R_0 and R_4 turned out to be the same region and therefore the edge f_1 can be pulled off the two crossings and isotoped around S with an extra crossing. This is impossible because D is a minimal crossing diagram. This completes the proof of Theorem 5.

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